$\boxed{\text{RECITATION 4}}$

- N - dimensional Gaussian distribution

Consider N Gaussian random variables $\overbrace{x_1, x_2, \dots, x_N}^{=\vec{x}}$. Their joint distribution is given by the following probability density:

$$\rho(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N \underbrace{\det C}_{>0 \text{ for positive define matrix}}}} \, e^{-\frac{1}{2}\sum_{i,j}\left[(x_i-\mu_i)(x_j-\mu_j)C_{ij}^{-1}\right]}$$

$x^T C x > 0 \quad \forall \vec{x} \neq 0$

where μ_i is the mean of variable x_i, C is the $N \times N$ positive-define and symmetric covariance matrix such that:

$C_{ii} > \sum_{j\neq i} |C_{ij}|$

$C_{ii} > 0$

$C_{ii} = \sigma_i^2$ diagonal elements

$C_{ij} = \langle (x_i-\mu_i)(x_j-\mu_j)\rangle = \langle x_i x_j\rangle_c$ off - diagonal elements

Characteristic function

$$\varphi_{\vec{x}}(\vec{k}) = \left\langle e^{-i\sum_i k_i x_i}\right\rangle = \frac{1}{\sqrt{(2\pi)^N \det C}} \int d^N\vec{x} \; e^{-i\vec{k}\cdot\vec{x} - \frac{1}{2}\underbrace{(\vec{x}-\vec{\mu})}_{\vec{y}}{}^T C^{-1}(\vec{x}-\vec{\mu})}$$

$$= \frac{1}{\sqrt{(2\pi)^N \det C}} \int d^N\vec{y} \; e^{-ik\cdot y - ik\cdot\mu} \, e^{-y^T C^{-1} y} = \frac{e^{-i\vec{k}\cdot\vec{\mu}}}{\sqrt{(2\pi)^N \det C}} \underbrace{\int d^N\vec{y} \; e^{-ik\,y - y^T C^{-1} y}}_{\text{standard gaussian integral}}$$

$$= \frac{e^{-i\vec{k}\,\vec{\mu}}}{\cancel{\sqrt{(2\pi)^N \det C}}} \cancel{\sqrt{(2\pi)^N \det C}} \; e^{-\frac{1}{2}\vec{k}^T C \vec{k}}$$

$$= e^{-i\vec{k}\cdot\vec{\mu} - \frac{1}{2}\vec{k}^T C \vec{k}}$$

$$\ln \varphi_{\vec{x}}(\vec{k}) = -i\vec{k}\,\vec{\mu} - \frac{1}{2}\vec{k}^T C \vec{k}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \langle x_i \rangle_c &= \mu_i \\ \langle x_i x_j \rangle_c &= C_{ij} \end{aligned}}$$

All higher-order cumulants vanish like in the 1-D case.

<u>Wick's theorem</u>: if the mean of all variables vanish, i.e. $\mu_i = 0 \ \forall i$, then the computation of the moments simplify. All odd moments vanish, all even moments can be written as a sum over all possible pairings of variables

ex. $\langle x_i x_j x_l x_m \rangle = \langle x_i x_j \rangle \langle x_l x_m \rangle + \langle x_i x_l \rangle \langle x_j x_m \rangle + \langle x_i x_m \rangle \langle x_j x_l \rangle$

$= C_{ij} C_{lm} + C_{il} C_{jm} + C_{im} C_{jl}$

how many pair partitions of four elements? In how many ways we can break it into <u>distinct</u> pairs

$$\frac{2m!}{(2^m \ m!)} \overset{m=2}{=} \frac{4!}{2^2 \ 2!} = \frac{24}{4 \times 2} = \frac{24}{8} = 3$$

2m elements → we want to write as the product of m pairs, so we there are m! way to permute m pairs, so we are overcounting by a factor m!.

$2m!$ total number of arrangements of 2m elements

$m! \ 2^m$ permutation of the elements in each pair

$m!$ → permutation of the pairs

• Time scale separation

how to describe relaxation to equilibrium?

$\tau_c \ll \tau_{MFP} \ll \tau_F$

we first made a coarse grain around this scale
↳ to make the collision term tractable

From Liouville:

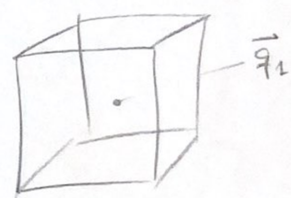
interactions among particles

$$\partial_t \hat{f}_1(\vec{q}_1, \vec{p}_1, t) + \{\hat{f}_1, H_1\} = \underbrace{\frac{\partial}{\partial t}\hat{f}\Big|_{coll}}_{\text{causes change } \sim \tau_{MFP}}$$

where $\hat{f}_1(\vec{q}_1, \vec{p}_1, t) = \frac{1}{V_c(\vec{q}_1)} \int_{V_c(\vec{q}_1)} d\vec{q}_1{}' \, f_1(\vec{q}_1{}', \vec{p}_1, t)$

coarse-grained number density

$$\{\hat{f}_1, H_1\} = \frac{\partial \hat{f}_1}{\partial \vec{q}_i} \cdot \frac{\partial H_1}{\partial \vec{p}_i} - \frac{\partial \hat{f}_1}{\partial \vec{p}_i} \cdot \frac{\partial H_1}{\partial \vec{q}_i}$$



$$H_1 = \sum_{i=1}^{N} \left(\frac{\vec{p}_i^{\,2}}{2m} + U(x_i) \right)$$

$\frac{\partial}{\partial t}\hat{f}\Big|_{coll}$ cause changes on the time scale $\sim \tau_{MFP}$

$\hat{f}_1(t) = S\left(\frac{t}{\tau_{MFP}}\right)$ such that $S(u)$ does not depend on τ_{MFP}

$S'(u) \sim O(\tau_{MFP}{}^0)$

$\hat{f}_1'(t) = \frac{1}{\tau_{MFP}} S'\left(\frac{t}{\tau_{MFP}}\right)$

All the dependence has been absorbed in the rescaling of t.

Using Taylor series:

$$\hat{f}_1(t+\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{\partial^n \hat{f}_1}{\partial t}\Big|_{\tau=0} = \hat{f}_1(t) + \frac{\tau}{\tau_{MFP}} S'\left(t/\tau_{MFP}\right) + \frac{1}{2}\frac{\tau^2}{\tau_{MFP}{}^2} S''\left(\frac{t}{\tau_{MFP}}\right) + \text{h.o.t.}$$

By choosing $\tau \ll \tau_{\text{MFP}}$ you are allowed to truncate the series.

$$\hat{f}_1(t+\tau) \simeq \hat{f}_1(t) + \frac{\tau}{\tau_{\text{MFP}}} S'(\tau/_{\text{MFP}})$$

$$\hat{f}_1(t+\tau) \simeq \hat{f}_1(t) + \tau \hat{f}_1'(\tau)$$

$$\Rightarrow \frac{\hat{f}_1(t+\tau) - \hat{f}_1(t)}{\tau} = \hat{f}_1'(\tau)$$

$$\left.\frac{\partial}{\partial t} f\right|_{coll} \simeq \frac{1}{\tau}\left(\hat{f}_1(\vec{q}_1, \vec{p}_1, t+\tau) - \hat{f}_1(\vec{q}_1, \vec{p}_1, t)\right)$$

$$\simeq \frac{1}{\tau V_c(\vec{q}_1)\,|d\vec{p}_1|} \underbrace{V_c(\vec{q}_1)\,|d\vec{p}_1|\left(\hat{f}_1(\vec{q}_1, \vec{p}_1, t+\tau) - \hat{f}_1(\vec{q}_1, \vec{p}_1, t)\right)}_{N^+(\tau, \vec{p}_1) \; - \; N^-(\tau, \vec{p}_1)}$$

$$N^-(\tau, \vec{p}_1) = \int d^3\vec{p}_2 \underbrace{d^2\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2')}_{\text{it is an integral on } \vec{p}_2', \vec{p}_1'} \hat{f}_1(\vec{q}, \vec{p}_1, t)\, \hat{f}_1(\vec{q}, \vec{p}_2, t)\, |\vec{v}_1 - \vec{v}_2| \times \tau \times |d\vec{p}_1| \times V_c(\vec{q})$$

\# of collisions
in the time interval
τ which "remove"
a particle with momentum
$\vec{p}_1$

check the dimensionality:

$$\cancel{p^3}\,\cancel{A}\;\frac{1}{\cancel{p^3 V}}\;\frac{1}{\cancel{p^3 V}}\;\frac{\cancel{L}}{\cancel{T}}\;\cancel{T}\;\cancel{p^3 V} \overset{ok}{=}$$

Similar computation for $N^+(\tau, \vec{p}_1)$

$$\partial_t \hat{f}_1 + \{\hat{f}_1, H_1\} = \int d^3\vec{p}_2 \, d\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2') \, |\vec{v}_2 - \vec{v}_1| \Big[\hat{f}_1(\vec{q}_1, \vec{p}_1', t)\hat{f}_1(\vec{q}_1, \vec{p}_2', t) - \hat{f}_1(\vec{q}_1, \vec{p}_1, t)\hat{f}_1(\vec{q}_1, \vec{p}_2, t)\Big]$$

Rewriting:

$$\boxed{\partial_t f + \underbrace{L_F f}_{\substack{\text{controls relaxation} \\ \sim \tau_F}} = \underbrace{C(f,f)}_{\substack{\text{controls relaxation} \\ \sim \tau_{MFP}}}}$$

Using the H-theorem we have seen that on the timescale $\sim \tau_{MFP}$ when $L_F f$ does not matter, a solution to this can be express in terms of collisional invariants such that

$$\boxed{\ln f_1 + \ln f_2 = \ln f_1' + \ln f_2'}$$

5 collisional invariants:

i) particle numbers
ii) total momentum
iii) energy

What about the long-term relaxation?

τ_F is the macroscopic time scale in the system

We rescale time $\hat{t} = \frac{\tau}{\tau_F}$ $\quad \tau = \hat{t}\,\tau_F$

this is why collisions still play a role

$\sim O(1) + \sim O(\varepsilon)$

$$\frac{1}{\tau_F}\partial_{\hat{t}} f + \underbrace{L_F f}_{\frac{1}{\tau_F}\hat{L}_F f} = \underbrace{C(f,f)}_{\frac{1}{\tau_{MFP}}\hat{C}(f,f)} \implies \partial_{\hat{t}} f + \hat{L}_F f = \underbrace{\frac{\tau_F}{\tau_{MFP}}}_{\frac{1}{\varepsilon}} \hat{C}(f,f) \quad \underline{f_1}$$

We have adimensionalize time and we find a small parameter $\varepsilon = \frac{\tau_{MFP}}{\tau_F}$ due to the time-scale separation

This suggests to solve the equation perturbatively in ε

$$\boxed{f = f_0 + \varepsilon f_1}$$

$O\left(\frac{1}{\varepsilon}\right)$: $\hat{C}(f_0, f_0) = 0 \Rightarrow f_0 = f^{LEQ}(\vec{q}, \vec{p}, t)$

it tells that f_0 can be parameterized by the collisional invariants! It also tells that on the timescale τ_F $\hat{C}(f, f_0)$ does not evolve to any longer.

$$\boxed{f_0(\vec{q}, p, t) = \frac{n(\vec{q}, t)}{(2\pi M k_B T)^{3/2}} e^{-\frac{(\vec{p} - M\vec{u})^2}{2M k_B T}}}$$

convenient way of parameterize it

$$\boxed{\frac{n(q,t)}{(2\pi M k_B T(q,t))^{3/2}} e^{-\frac{M \delta \vec{v}^2}{2 k_B T}}}$$

with

density field

$n(\vec{q}, t) = \int d^3\vec{p}\, f_0$

$n\vec{u}(\vec{q}, t) = \int d^3p\, \vec{v}\, f_0$

velocity field

$n \frac{3}{2} k_B T(\vec{q}, t) = \int d^3\vec{p}\, (\vec{v} - u)^2 \frac{M}{2} f_0$

temperature field

6

We have derived in class the <u>hydrodynamic equations</u>

For these fields using the Boltzmann equation.

Let's redo together the derivation of the evolution of $\vec{u}$

$$u_\alpha = \frac{1}{n(q,t)} \int d^3\vec{p}\, v_\alpha f(\vec{q},\vec{p},t)$$

$$\partial_t u_\alpha = -\frac{1}{n^2}\partial_t n \underbrace{\int d^3p\, v_\alpha f}_{n u_\alpha} + \frac{1}{n(\vec{q},t)} \int d^3\vec{p}\, v_\alpha (\partial_t f)$$ using Boltzmann

$$= -\partial_t n \frac{u_\alpha}{n} + \frac{1}{n} \underbrace{\int d^3\vec{p}\, v_\alpha \left[-v_\beta \cdot \frac{\partial}{\partial q_\beta} f\right]}_{\text{from Poisson bracket}} + \frac{1}{n}\int d^3\vec{p_1} \int d^3\vec{p_2}\, d^2\sigma(\vec{p_1}',\vec{p_2}' \to \vec{p_1},\vec{p_2})\, v_\alpha{}^1$$
$$|\vec{v}_1 - \vec{v}_2| (f_1' f_2' - f_1 f_2)$$

$$= -\frac{u_\alpha}{n}\partial_t n - \frac{1}{n}\partial_{q_\beta}(n\langle v_\alpha v_\beta\rangle) + \underbrace{\frac{1}{n}\int d^3\vec{p_1}\, d^3\vec{p_2}\, d^2\sigma(\vec{p_1}',\vec{p_2}' \to \vec{p_1},\vec{p_2})|\vec{v}_1-\vec{v}_2|}_{①} \, (\underbrace{f_1' f_2'}_{\substack{\text{source term}\\ \text{for } \vec{p_1},\vec{p_2}}} - \underbrace{f_1 f_2}_{\substack{\text{sink term for}\\ \vec{p_1},\vec{p_2}}}) v_\alpha{}^1$$

① because $\vec{p_1}, \vec{p_2}$ are dummy variables $$\frac{1}{2n}\int d^3\vec{p_1}\, d^3p_2\, d^2\sigma(\vec{p_1}',\vec{p_2}' \to \vec{p_1},\vec{p_2})|\vec{v}_1-\vec{v}_2|(f_1'f_2' - f_1 f_2)(v_\alpha{}^1 + v_\alpha{}^2)$$

here you are labelling $\vec{p_1}', \vec{p_2}'$ the preimage (of $\vec{p_1}, \vec{p_2}$)
but noone prevents you from calling $\vec{p_1}, \vec{p_2}$ the preimage

$$= \frac{1}{2n}\int d^3\vec{p_1}'\, d^3\vec{p_2}'\, d^2\sigma(\vec{p_1},\vec{p_2} \to \vec{p_1}',\vec{p_2}')|\vec{v}_1'-\vec{v}_2'|(\underbrace{f_1 f_2}_{\substack{\text{source}\\ \text{of } \vec{p_1}',\vec{p_2}'}} - \underbrace{f_1' f_2'}_{\substack{\text{source}\\ \text{of } \vec{p_1}\,\vec{p_2}}})(v_\alpha^{1\,\prime} + v_\alpha^{2\,\prime})$$

but now, because of elastic collisions

$$d^3\vec{p_1}'\, d^3\vec{p_2}' = J\, d^3\vec{p_{cm}}'\, d^3\vec{p_D}' = J\, d^3\vec{p_{cm}}\, d^3\vec{p_D} = \cancel{\frac{J}{J}}\, d^3\vec{p_1}\, d^3\vec{p_2}$$

$$d^2\sigma(\vec p_1, \vec p_2 \to \vec p_1', \vec p_2') = d^2\sigma(\vec p_1', \vec p_2' \to \vec p_1, \vec p_2)$$

$$|\vec v_1' - \vec v_2'| = |\vec v_1 - \vec v_2|$$

$$= \underbrace{\frac{1}{4n}\int d^3\vec p_1\, d^3\vec p_2\, d^2\sigma(\vec p_1', \vec p_2' \to \vec p_1, \vec p_2)\, |\vec v_1 - \vec v_2|\, (f_1' f_2' - f_1 f_2)}_{} \cancel{\left(v_\alpha^1 + v_\alpha^2 - v_\alpha^{1'} - v_\alpha^{2'}\right)}$$

so the collision term vanish!

but $\vec p_{cm} = \frac{\vec p_1 + \vec p_2}{2} = \frac{\vec p_1' + \vec p_2'}{2}$

$$\underline{v_\alpha^1 + v_\alpha^2 = v_\alpha^{1'} + v_\alpha^{2'}}$$

Same trick as in the H-theorem!

Actually, this is what we expect!

<u>Slow modes</u> evolves because of transport, they are collisional invariants.

$$= -\frac{u_\alpha}{n}\underbrace{\partial_t n}_{} - \frac{1}{n}\partial_{q_\beta}(n\langle v_\alpha v_\beta\rangle) + 0$$

using the first hydrodynamic equation

$$= \frac{u_\alpha}{n}\partial_{q_\beta}(n u_\beta) - \frac{1}{n}\partial_{q_\beta}(n\langle v_\alpha v_\beta\rangle)$$

$$\Rightarrow \underbrace{\partial_t u_\alpha + u_\beta \partial_{q_\beta} u_\alpha}_{D_t u_\alpha} = \underbrace{u_\beta \partial_{q_\beta} u_\alpha + u_\alpha \partial_{q_\beta} u_\beta}_{\partial_{q_\beta}(u_\alpha u_\beta)} + \frac{u_\alpha u_\beta}{n}\partial_{q_\beta} n - \frac{1}{n}\partial_{q_\beta}(n\langle v_\alpha v_\beta\rangle)$$

$$D_t u_\alpha = \underbrace{\partial_{q_\beta}(u_\alpha u_\beta) + \frac{u_\alpha u_\beta}{n}\partial_{q_\beta} n}_{\frac{1}{n}\partial_{q_\beta}(u_\alpha u_\beta n)} - \frac{1}{n}\partial_{q_\beta}(n\langle v_\alpha v_\beta\rangle)$$

$$= -\frac{1}{n}\partial_{q_\beta}\left(n\langle v_\alpha v_\beta\rangle - \langle v_\alpha\rangle\langle v_\beta\rangle n\right)$$

$$D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta}\left(n\langle\delta v_\alpha \delta v_\beta\rangle_f\right)$$

where $\delta\vec{v} = \vec{v} - \langle\vec{v}\rangle_f$

$$\boxed{M D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta} P_{\alpha\beta}}$$

with $P_{\alpha\beta} = Mn\langle\delta v_\alpha \delta v_\beta\rangle_f$

$$\boxed{D_t n(\vec{q},t) = -n\,\partial_{q_\alpha} u_\alpha}$$

$$\boxed{\partial_t T + u_\alpha \partial_{q_\alpha} T = -\frac{2}{3nk_B}\partial_\alpha h_\alpha - \frac{2}{3nk_B}P_{\alpha\beta}u_{\alpha\beta}}$$

with $h_\alpha = \frac{nM}{2}\langle\delta v_\alpha \delta\vec{v}\rangle_f$ heat flux along α

$u_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right)$ strain rate tensor

these are exact but not close!

We have seen in class how to compute them to zeroth order in ε $\quad (f = f_0 + \varepsilon \cancel{f_1})$

$$\boxed{\begin{aligned} h_\alpha^0 &= 0 \\ P_{\alpha\beta}^0 &= n k_B T \delta_{\alpha\beta} \end{aligned}}$$

How to compute them to first order? → we go to higher order in the perturbation theory

$O(1)$: $\partial_t^* f_0 + L_F f_0 = \hat{C}(f_0, f_1) + \hat{C}(f_1, f_0)$

(single time approximation

$\boxed{\partial_t^* f_0 + L_F f_0 = -f_1}$ extra ingredient: single time approximation

$$f = f_0 + \varepsilon f_1 = f_0\Big(1 - \overbrace{\tau_{MFP}\left(\partial_t + \frac{\vec{p}}{M}\cdot\partial_{\vec{q}}\right)\ln f_0}^{g_1}\Big)$$

$$\Longrightarrow \quad g_1 = -\frac{\tau_{MFP} M}{k_B T}\, u_{\alpha\beta}\left(\delta v_\alpha \delta v_\beta - \frac{\delta \bar{v}^2}{3}\delta_{\alpha\beta}\right) - \frac{\tau_{MFP}}{T}\,\delta v_\alpha \left(\partial_{q_\alpha} T\right)\left(\frac{M}{2k_BT}\delta \bar{v}^2 - \frac{5}{2}\right)$$

you have used the zeroth order hydrodynamic equations here

$f_0(1+g_1)$

Pressure:

$$P_{\alpha\beta} = nM\langle \delta v_\alpha \delta v_\beta\rangle_f = \underbrace{nM \langle \delta v_\alpha \delta v_\beta\rangle_0}_{n k_B T \delta_{\alpha\beta}} + nM \langle \delta v_\alpha \delta v_\beta\, g_1\rangle_0$$

$$= -\frac{M^2 \tau_{MFP}}{k_B T}\, u_{\gamma\delta}\left(\underbrace{\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\delta\rangle_0}_{(A)} - \underbrace{\frac{\delta_{\gamma\delta}}{3}\langle \delta \bar{v}^2 \delta v_\alpha \delta v_\beta\rangle_0}_{(B)}\right)$$

Using Wick's theorem

$$\frac{2m!}{2^m m!} = \frac{4!}{4\,2!} = \frac{24}{8} = 3$$

$$\frac{2m!}{2^m m!} = \frac{2}{2} = 1$$

(A) $$u_{\gamma\delta}\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\delta\rangle_0 = \sum_{\delta,\gamma}\Big[u_{\gamma\delta}\overbrace{\langle \delta v_\alpha \delta v_\beta\rangle}^{\delta_{\alpha\beta}}\overbrace{\langle \delta v_\gamma \delta v_\delta\rangle}^{\delta_{\gamma\delta}} + u_{\gamma\delta}\overbrace{\langle \delta v_\alpha \delta v_\gamma\rangle}^{\delta_{\alpha\gamma}}\overbrace{\langle \delta v_\beta \delta v_\delta\rangle_0}^{\delta_{\beta\delta}} + u_{\gamma\delta}\overbrace{\langle \delta v_\alpha \delta v_\delta\rangle}^{\delta_{\alpha\delta}}\overbrace{\langle \delta v_\beta \delta v_\gamma\rangle}^{\delta_{\beta\gamma}}\Big] = \left(3u_{\gamma\gamma}\delta_{\alpha\beta} + u_{\alpha\beta} + u_{\alpha\beta}\right)\left(\frac{k_BT}{M}\right)^2$$

divergence $= \partial_\gamma u_\gamma$

(B) $$u_{\gamma\delta}\frac{\delta_{\gamma\delta}}{3}\langle \delta v_\alpha \delta v_\beta \delta v_\varepsilon \delta v_\varepsilon\rangle = \sum_{\delta,\gamma,\varepsilon} u_{\gamma\delta}\frac{\delta_{\gamma\delta}}{3}\langle \delta v_\alpha \delta v_\beta \delta v_\varepsilon \delta v_\varepsilon\rangle =$$

$$= \frac{\overbrace{3u_{\gamma\gamma}}^{\partial_\gamma u_\gamma}}{3}\left[\sum_\varepsilon \langle \delta v_\alpha \delta v_\beta \delta v_\varepsilon \delta v_\varepsilon\rangle\right] = \frac{\partial_\gamma u_\gamma}{3}\Big[\sum_\varepsilon \overbrace{\langle \delta v_\alpha \delta v_\beta\rangle}^{\delta_{\alpha\beta}}\overbrace{\langle \delta v_\varepsilon \delta v_\varepsilon\rangle}^{\delta_{\varepsilon\varepsilon}} +$$

$$+\sum_{\varepsilon} \underbrace{\langle \delta v_\alpha \delta v_\varepsilon\rangle}^{\delta_{\alpha\varepsilon}} \underbrace{\langle \delta v_\beta \delta v_\varepsilon\rangle}^{\delta_{\beta\varepsilon}} + \sum_{\varepsilon} \underbrace{\langle \delta v_\alpha \delta v_\varepsilon\rangle}^{\delta_{\alpha\varepsilon}} \underbrace{\langle \delta v_\beta \delta v_\varepsilon\rangle}^{\delta_{\beta\varepsilon}} \Big]$$

$$= \frac{\partial_\gamma u_\gamma}{3} \left(\frac{k_B T}{n}\right)^2 n \Big[\delta_{\alpha\beta}\, 3 + \delta_{\alpha\beta} + \delta_{\alpha\beta} \Big] = \frac{\partial_\gamma u_\gamma}{3}\, 5 \left(\frac{k_B T}{n}\right)^2 \delta_{\alpha\beta}$$

All in all:

$$P_{\alpha\beta} = n k_B T \delta_{\alpha\beta} - n k_B T\, \tau_{MFP} \left(\partial_\gamma u_\gamma \delta_{\alpha\beta} + 2 u_{\alpha\beta} - \frac{5}{3} \partial_\gamma u_\gamma \delta_{\alpha\beta} \right) \underline{\underline{ok}}$$

<u>Heath flow:</u>

$$h_\alpha = \frac{n\Pi}{2} \langle \delta v_\alpha \delta \bar{v}^2 (1 + g_1) \rangle_0 = \frac{n\Pi}{2} \cancel{\langle \delta v_\alpha \delta \bar{v}^2 \rangle_0}^{\,0} + \frac{n\Pi}{2} \langle \delta v_\alpha \delta \bar{v}^2 g_1 \rangle_0$$

$$= \frac{n M^2}{4 k_B T} \left(-\frac{\tau_{MFP}}{T} \right) (\partial_{q_\beta} T) \underbrace{\langle \delta v_\alpha \delta \bar{v}^2 \delta v_\beta \delta \bar{v}^2 \rangle}_{A} + \frac{n\Pi}{4}\, 5\, \frac{\tau_{MFP}}{T} (\partial_{q_\beta} T) \underbrace{\langle \delta v_\alpha \delta v_\beta \delta \bar{v}^2 \rangle}_{B}$$

(A) $\langle \delta v_\alpha \delta v_\varepsilon \delta v_\varepsilon \delta v_\beta \delta v_\gamma \delta v_\gamma \rangle = \sum_{\varepsilon,\gamma} \delta_{\alpha\beta} \langle \delta v_\alpha^2 \delta v_\varepsilon^2 \delta v_\gamma^2 \rangle$

$$\frac{2m!}{2^m m!} = \frac{6!}{2^3 3!} = \frac{720}{8 \cdot 6} = 15 \qquad \frac{2m!}{2^m m!} = \frac{24}{4 \cdot 2} = 3$$

$$= \delta_{\alpha\beta} \langle \delta v_\alpha^6 \rangle + 2 \sum_{\varepsilon \neq \alpha} \delta_{\alpha\beta} \underbrace{\langle \delta v_\alpha^4 \delta v_\varepsilon^2 \rangle_{\gamma=\alpha}}_{2\times 2\times 3} + \sum_{\varepsilon\neq\alpha} \delta_{\alpha\beta} \underbrace{\langle \delta v_\alpha^2 \delta v_\varepsilon^4 \rangle_{\varepsilon=\gamma}}_{2\times 3}$$

combinatorics

Coordinate

$$+ \sum_{\varepsilon\neq\alpha\neq\gamma} \delta_{\alpha\beta} \langle \delta v_\alpha^2 \delta v_\varepsilon^2 \delta v_\gamma^2 \rangle =$$

$$= \delta_{\alpha\beta} \Big[15 + 12 + 6 + 2 \Big] \left(\frac{k_B T}{n}\right)^3$$

$$= \delta_{\alpha\beta}\, 35 \left(\frac{k_B T}{n}\right)^3$$

(B) $\dfrac{nm\,5}{4}\left(\dfrac{\tau_{mFP}}{T}\right)\left(\partial_{q_\beta} T(q,t)\right)\left\langle \delta v_\alpha \delta v_\beta \delta \vec{v}^{\,2}\right\rangle.$

$$\sum_\gamma \langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\gamma\rangle = \sum_\gamma \Big[\underbrace{\langle \delta v_\alpha \delta v_\beta\rangle}_{\delta_{\alpha\beta}} \underbrace{\langle \delta v_\gamma \delta v_\gamma\rangle}_{\delta_{\gamma\gamma}} + \underbrace{\langle \delta v_\alpha \delta v_\gamma\rangle}_{\delta_{\alpha\gamma}} \underbrace{\langle \delta v_\beta \delta v_\gamma\rangle}_{\delta_{\gamma\beta}}$$

$$+ \underbrace{\langle \delta v_\alpha \delta v_\gamma\rangle}_{\delta_{\alpha\gamma}} \underbrace{\langle \delta v_\beta \delta v_\gamma\rangle}_{\delta_{\beta\gamma}} \Big] = \Big[3\,\delta_{\alpha\beta} + 2\,\delta_{\alpha\beta}\Big] \left(\frac{k_B T}{m}\right)^2$$

All in all:

$$h_\alpha = \frac{n m^2}{k_B T\, 4}\left(-\frac{\tau_{mFP}}{T}\right)\left(\partial_{q_\beta} T\right) 35\, \delta_{\alpha\beta} \left(\frac{k_B T}{m}\right)^3 + 5\left(\frac{\tau_{mFP}}{T}\right)\left(\partial_{q_\beta} T\right) 5\, \delta_{\alpha\beta} \left(\frac{k_B T}{m}\right)^2$$

$$= -\frac{n m}{4T}\tau_{mFP}\left(\partial_{q_\beta} T\right)\delta_{\alpha\beta}\left[\frac{m}{\cancel{k_B T}}\, 35 \left(\frac{k_B T}{m}\right)^{\cancel{3}\,2} - 25\left(\frac{k_B T}{m}\right)^2\right]$$

$$= -\frac{n m}{4T}\tau_{mFP}\left(\partial_{q_\alpha} T\right) 10 \left(\frac{k_B T}{m}\right)^2 = -\underbrace{\frac{5}{2}\, n\, \frac{\tau_{mFP}\, k_B^2\, T}{m}}_{\text{thermal conductivity}}\left(\partial_{q_\alpha} T\right).$$